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CONTACT PROBLEMS OF ELASTICITY THEORY FOR WEDGE-SHAPED REGIONS UNDER CONDITIONS OF FRICTION AND ADHESION†

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A wedge-shaped punch with included angle close to π is pressed onto an elastic half-plane by a centrally applied vertical force P; the contact area, divided into a frictional region and an adhesive region, is either known in advance (problem 1a) or has to be determined (problem 1b). Two-dimensional contact is investigated for an elastic wedge-shaped punch pressed down by a vertical force P, a horizontal force T and a couple of moment M (problem 2); the punch extends beyond the apex of the wedge and is flat-faced; the contact area is divided into an inner adhesive region and two outer regions of Coulomb friction.

An analytical solution, accurate to within any prescribed limits, will be presented for these problems, thus generalizing the solution described in [1]; the method used is that employed in [2], where the problem is reduced to a Riemann vector problem for two pairs of functions (problems 1a, 1b) or three pairs (problem 2), which is then solved. The boundaries of the adhesive and frictional regions will be determined, and in problem 1b the contact area also. Formulae will be developed for the contact stresses. It will be shown that the stresses are continuous across the common boundary of the adhesive and frictional regions. The statement made in [3] that when the punch is pressed symmetrically onto the half-plane the ratio λ of the length $2b$ of the adhesive region to the length $2a$ of the contact region is the same for a flat-faced punch and a punch whose profile is described by the function $f(x) = \Lambda|x|^n$ $(n \geqslant 1)$ will be disproved. It will be proved that if the punch profile is smooth in the vicinity of the point a, then λ is uniquely defined by Poisson's ratio ν, the coefficient of friction μ and the exponent n; it is independent of the coefficient Λ and the force P (in particular, λ in problem 1b is independent of the included angle of the punch).

The introduction of the regions of friction in the contact area for problem 2, enables one not only to eliminate oscillation of the contact stresses near the ends of the punch, but also to construct an analytic solution of the contact problem for a wedge when the contact shear and normal stresses are unknown (such a solution has not been obtained when the punch is fully adhesive).

The problem of two wedge-shaped elastic bodies in contact with no shear stresses was solved in [4].

1. A WEDGE-SHAPED PUNCH

CONSIDER a wedge-shaped punch, pressed down onto an elastic half-plane $(0<r<\infty,\ -\pi<\theta<0)$ with Poisson's ratio ν and modulus of elasticity E; the punch, with an included angle 2γ of nearly π, is pressed down by a centrally applied vertical force P. The contact area $(0<r<a,\ \theta=-\pi$ and $\theta=0)$ is divided into zones of adhesion $(0<r<b,\ \theta=-\pi$ and $\theta=0)$ and Coulomb friction $(b<r<a,\ \theta=-\pi$ and $\theta=0)$

$$u_\theta = r\operatorname{ctg}\gamma + \delta_n \qquad (0<r<a,\ \ \theta=-\pi,\ \ \theta=0)$$
$$u_r = 0 \quad (0<r<b,\ \ \theta=-\pi,\ \theta=0)$$
$$\tau_{r\theta} - \mu\sigma_\theta = 0 \quad (b<r<a,\ \ \theta=-\pi); \quad \tau_{r\theta}+\mu\sigma_\theta = 0 \quad (b<r<a,\ \ \theta=0)$$

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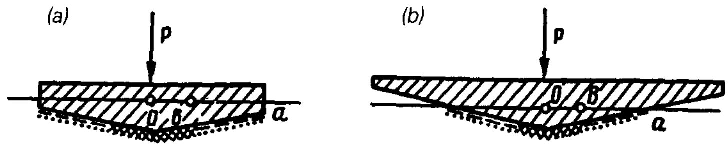

FIG. 1.

(δ_n is an additive constant and μ is the coefficient of friction). Outside the contact area the boundary of the half-plane is free from stresses. The quantity a is either fixed (problem 1a—Fig. 1a) or has to be determined (problem 1b—Fig. 1b). The shear stresses in the adhesive region are too small to create slipping: $|\tau_{r\theta}|<\mu|\sigma_\theta|$. The normal stresses must be negative.

Symmetry dictates that problems 1a, 1b reduce to a two-dimensional boundary-value problem for a quarter-plane, with boundary conditions

$$\begin{gathered}\sigma_\theta|_{\theta=0}=\chi_1(r),\quad (\tau_{r\theta}+\mu\sigma_\theta)_{\theta=0}=\chi_2(r),\\ \left.\frac{\partial u_r}{\partial r}\right|_{\theta=0}=\nu_*\psi_1(r),\quad \left.\frac{\partial u_\theta}{\partial r}\right|_{\theta=0}=\nu_*\psi_2(r)+\operatorname{ctg}\gamma,\quad \nu_*=\frac{1+\nu}{E}\\ u_\theta|_{\theta=-\pi/2}=0,\quad \tau_{r\theta}|_{\theta=-\pi/2}=0\quad (0<r<\infty)\end{gathered}\tag{1.1}$$

where $\chi_j(r)$ and $\psi_j(r)$ $(j=1,2)$ are unknown functions such that

$$\operatorname{supp}\chi_1\subset[0,a],\quad \operatorname{supp}\chi_2\subset[0,b]$$
$$\operatorname{supp}\psi_1\subset[b,\infty),\quad \operatorname{supp}\psi_2\subset[a,\infty)$$

Since the punch is in equilibrium

$$\int_0^a \chi_1(r)\,dr=-\frac{P}{2}\tag{1.2}$$

Denote the Mellin transforms of the unknown functions by

$$\begin{aligned}\Phi_1^-(s)&=\int_0^1\chi_1(a\rho)\rho^s d\rho,\quad \Phi_2^-(s)=\int_0^1\chi_2(b\rho)\rho^s d\rho\\ \Phi_1^+(s)&=\int_1^\infty\psi_1(b\rho)\rho^s d\rho,\quad \Phi_2^+(s)=\int_1^\infty\psi_2(a\rho)\rho^s d\rho\end{aligned}\tag{1.3}$$

The functions $\Phi_j^\pm(s)$ are analytic in the domain $D^\pm$: $\operatorname{Re}(s)\lessgtr\gamma\in(-1,0)$ and satisfy the following inhomogeneous Riemann matrix problem [2]

$$\begin{gathered}\lambda^{s+1}\Phi_1^+(s)=K_1(s)\Phi_1^-(s)-\kappa_+\lambda^{s+1}\operatorname{tg}\tfrac12\pi s\Phi_2^-(s)\\ (s+1)^{-1}\nu_0+\Phi_2^+(s)=K_0(s)\Phi_1^-(s)-\kappa_-\lambda^{s+1}\Phi_2^-(s),\qquad s\in\Gamma:\ \operatorname{Re}(s)=\gamma\\ K_1(s)=\kappa_-+\mu\kappa_+\operatorname{tg}\tfrac12\pi s,\quad K_0(s)=\kappa_+\operatorname{ctg}\tfrac12\pi s+\mu\kappa_-\\ \kappa_\pm=\tfrac12(\kappa\pm1),\quad \kappa=3-4\nu,\quad \nu_0=\nu_*^{-1}\operatorname{ctg}\gamma,\qquad \lambda=a^{-1}b\end{gathered}\tag{1.4}$$

which, after factorizing $K_0(s)$, we can write in the form

$$\begin{gathered}\kappa_*[K_0^-(s)]^{-1}\Phi_2^-(s)=K_0^+(s)\Phi_1^+(s)-\lambda^{-s-1}[K_0^-(s)]^{-1}K_1(s)\,[\Phi_2^+(s)+(s+1)^{-1}\nu_0]\\ [K_0^+(s)]^{-1}[\Phi_2^+(s)+(s+1)^{-1}\nu_0]=K_0^-(s)\Phi_1^-(s)-\kappa_-\lambda^{s+1}[K_0^+(s)]^{-1}\Phi_2^-(s),\quad s\in\Gamma\\ K_0^+(s)=-\frac{\kappa_0\Gamma(-\tfrac12 s)}{\Gamma(1-\tfrac12 s-\alpha)},\quad K_0^-(s)=\frac{\Gamma(1+\tfrac12 s)}{\Gamma(\alpha+\tfrac12 s)},\quad \kappa_0=\frac{\kappa_+}{\sin\pi\alpha}\\ \kappa_*=\kappa_-^2-\kappa_+^2,\quad \alpha=\pi^{-1}\operatorname{arcctg}(\mu\kappa_-\kappa_+^{-1})\end{gathered}$$

Noting that the function

$$\omega(s) = \nu_0 \lambda^{-s-1} [K_0^-(s)(s+1)]^{-1} K_1(s)$$

has a pole of first order at $s = -1$ ($\in D^+$), we obtain the following representation in the neighbourhood of that point

$$\omega(s) = a_0(s+1)^{-2} + (a_1 - a_0 \ln \lambda)(s+1)^{-1} + O(1), \quad s \to -1$$

$$a_0 = -2\mu\kappa_+ \nu_0 \pi^{-3/2} \Gamma(\alpha - ½)$$

$$a_1 = \nu_0 \pi^{-½} \Gamma(\alpha - ½) \{\kappa_- - \mu\kappa_+ \pi^{-1} [\psi(\alpha - ½) - \psi(½)]\}$$

($\psi(x)$ is the psi-function). Proceeding as in [2], we obtain the following formulae for the solution of the Riemann problem (1.4)

$$\begin{aligned}
&\Phi_1^-(s) = [K_0^-(s)]^{-1} Z_1(s) + [\kappa_* K_0^+(s)]^{-1} \kappa_- \lambda^{s+1} Z_2(s) \\
&\Phi_1^+(s) = [K_0^-(s)]^{-1} K_1(s) \lambda^{-s-1} Z_1(s) + [K_0^+(s)]^{-1} Z_2(s) \\
&\Phi_2^-(s) = \kappa_*^{-1} K_0^-(s) Z_2(s), \quad \Phi_2^+(s) = -\nu_0(s+1)^{-1} + K_0^+(s) Z_1(s) \\
&Z_1(s) = C + (s+1)^{-1} a_2 + \Psi_0^+(s), \quad a_2 = -\nu_0 \kappa_0^{-1} \pi^{-½} \Gamma(3/2 - \alpha) \\
&Z_2(s) = -a_0(s+1)^{-2} + (a_0 \ln \lambda - a_1)(s+1)^{-1} + \Psi_0^-(s) + \Psi_1^-(s)
\end{aligned} \tag{1.5}$$

where C is an arbitrary constant, $\Psi_0^\pm(s)$, $\Psi_1^-(s)$ are functions analytic in the domains $D^\pm$ and having the form

$$\Psi_0^\pm(s) = \sum_{j=0}^{\infty} \frac{A_j^\pm}{s + 2\alpha \mp 2j \mp 1 - 1}, \quad \Psi_1^-(s) = \sum_{j=0}^{\infty} \frac{B_j}{s + 1 + 2j} \tag{1.6}$$

The coefficients $A_j^\mp$, B_j have to be determined; their asymptotic behaviour is described by the following estimates [2]

$$A_j^+ = O(\lambda^{2j} j^{1-2\alpha}), \quad A_j^- = O(B_j) = O(\lambda^{2j} j^{-2+2\alpha}), \quad j \to \infty$$

The functions $\Phi_1^\mp(s)$ are analytic in $D^\pm$ if and only if the coefficient $A_j^\mp$, B_j satisfy the following infinite algebraic system of normal type

$$\begin{aligned}
&A_n^- = \lambda^{2n+2\alpha-1} \delta_{0n}^+ \left(C - \frac{a_2}{2n + 2\alpha - 1} - \sum_{j=0}^{\infty} \frac{A_j^+}{2n + 2j + 2} \right) \\
&B_n = \lambda^{2n} \delta_{1n}^+ \left(C + q_* \delta_{n0} - \sum_{j=0}^{\infty} \frac{A_j^+}{2n + 2j + 3 - 2\alpha} \right) \\
&A_n^+ = \lambda^{2n+3-2\alpha} \delta_{0n}^- \left(q_n + \sum_{j=0}^{\infty} \frac{A_j^-}{2n + 2j + 2} + \sum_{j=0}^{\infty} \frac{B_j}{2n + 2j + 3 - 2\alpha} \right)
\end{aligned} \tag{1.7}$$

where

$$\begin{aligned}
&q_* = -½ \pi^{-1/2} \mu \nu_0 \kappa_- \kappa_0^{-2} \sec \pi\alpha \Gamma(3/2 - \alpha) [\psi(3/2 - \alpha) - \psi(½)] \\
&q_n = -a_0(3 + 2n - 2\alpha)^{-2} + (a_0 \ln \lambda - a_1)(3 + 2n - 2\alpha)^{-1} \\
&\delta_{0n}^+ = 2\kappa_+ \kappa_* \Gamma^2(n + \alpha)(\pi\kappa_- n!^2)^{-1} \\
&\delta_{1n}^+ = 2\kappa_+ \kappa_0^2 \Gamma^2(n + ½) [\pi\kappa_- \Gamma^2(n + 3/2 - \alpha)]^{-1} \\
&\delta_{0n}^- = 2\kappa_+ \kappa_- \Gamma^2(n + 2 - \alpha)(\pi\kappa_* \kappa_0^2 n!^2)^{-1}
\end{aligned}$$

(δ_{mn} is the Kronecker delta). Expressing the unknown coefficients $A_n^\pm$, B_n successively as

$$\begin{aligned}
&A_n^\pm = C A_{n0}^\pm + A_{n1}^\pm, \quad B_n = C B_{n0} + B_{n1} \\
&A_{ni}^- = \lambda^{2n+2\alpha-1} \sum_{j=0}^{\infty} a_{nji}^- \lambda^{2j}, \quad A_{ni}^+ = \lambda^{2n+2} \sum_{j=0}^{\infty} a_{nji}^+ \lambda^{2j} \\
&B_{ni} = \lambda^{2n} \sum_{j=0}^{\infty} b_{nji} \lambda^{2j}
\end{aligned} \tag{1.8}$$

and introducing the notation

$$f_{n0}^- = f_{n0} = 1,\ f_{n0}^+ = 0,\ f_{n1}^- = -a_2(2n+2\alpha-1)^{-1},\quad f_{n1} = q_* \delta_{n0},\quad f_{n1}^+ = q_n$$

we can express system (1.7) in terms of recurrence relations

$$a_{n0i}^- = \delta_{0n}^+ f_{ni}^-,\quad b_{n0i} = \delta_{1n}^+ f_{ni}$$

$$a_{nki}^- = -\delta_{0n}^+ \sum_{j=1}^{k} \frac{a_{j-1,k-j,i}^+}{2(n+j)},\quad b_{nki} = -\delta_{1n}^+ \sum_{j=1}^{k} \frac{a_{j-1,k-j,i}^+}{2n+2j+1-2\alpha}$$

$$a_{n,k-1,i}^+ = \delta_{0n}^- [\lambda^{1-2\alpha}\delta_{k1} f_{ni}^+ + \sum_{j=1}^{k} (\frac{a_{j-1,k-j,i}^-}{2(n+j)} + \frac{\lambda^{1-2\alpha} b_{j-1,k-j,i}}{2n+2j+1-2\alpha})]$$

$$(n=0,1,\dots;\quad k=1,2,\dots;\quad i=0,1)$$

We now determine the constant C, the position of the point b and, for problem 1b, that of point a. By the equilibrium condition for the punch (1.2), as well as relationships (1.3), (1.5), (1.6) and (1.8), we find

$$C = -\frac{1}{1+\omega_0}\left(\frac{P}{2a\Gamma(\alpha)} + a_2 + \omega_1\right),\qquad \omega_i = \sum_{j=0}^{\infty} \frac{A_{ji}^+}{2(\alpha-1-j)}\quad (i=0,1) \tag{1.9}$$

The previously unknown position of b is found from the condition that the contact stresses are bounded there. We introduce the stress intensity factor as

$$K_b = \lim_{r\to b-0} (b-r)^{1-\alpha}(\tau_{r\theta} + \mu\sigma_\theta)_{\theta=0}$$

or, by (1.5) and an Abelian-type theorem

$$K_b = \frac{1}{\kappa_* \Gamma(\alpha)}\left(\frac{b}{2}\right)^{1-\alpha} \Omega(\lambda),\qquad \Omega(\lambda) = a_0 \ln\lambda - a_1 + \sum_{j=0}^{\infty} (A_j^- + B_j)$$

which implies the following transcendental equation for λ

$$\Omega(\lambda) = 0 \tag{1.10}$$

In problem 1a, when the position of a is known, Eq. (1.10) also determines the quantity $b = \lambda a$. But in problem 1b we must also find a from the condition that the normal contact stress σ_b be bounded in the vicinity of a (under this condition the shear stresses will also be bounded, because $\tau_{r\theta} = -\mu\sigma_\theta$ for $b<r<a$, $\theta=0$). Let

$$L_a = \lim_{r\to a-0} (a-r)^\alpha \sigma_\theta(r,0)$$

By (1.5) and an Abelian-type theorem, we find

$$L_a = 2^{1-\alpha} a^\alpha [\Gamma(1-\alpha)]^{-1} C$$

It follows from the condition $L_a = 0$ and from (1.9) that in problem 1b

$$C = 0,\quad a = -P[2\Gamma(\alpha)(a_2+\omega_1)]^{-1} \tag{1.11}$$

2. CONTACT STRESSES. ANALYSIS OF THE SOLUTION

We shall construct formulae for the contact stresses. Using inverse Mellin transforms in problem 1a, we deduce from (1.5) [in problem 1b, C and a are determined from (1.11)]

$$\sigma_\theta(r,0)=\chi_1(r)=I_1(r)+\kappa_*^{-1}\kappa_- I_2(r),\qquad 0<r<a$$
$$\tau_{r\theta}(r,0)=-\mu\chi_1(r),\quad b<r<a;\quad \tau_{r\theta}(r,0)=\chi_2(r)-\mu\chi_1(r),\quad 0<r<b$$
$$I_1(r)=\frac{1}{2\pi i}\int_\Gamma \frac{Z_1(s)}{K_0^-(s)}\left(\frac{r}{a}\right)^{-s-1}ds,\quad I_2(r)=\frac{1}{2\pi i}\int_\Gamma \frac{Z_2(s)}{K_0^+(s)}\left(\frac{r}{b}\right)^{-s-1}ds \tag{2.1}$$
$$\chi_2(r)=\frac{1}{2\pi i\kappa_*}\int_\Gamma K_0^-(s)Z_2(s)\left(\frac{r}{b}\right)^{-s-1}ds$$

Using the theory of residue and the equality

$$Z_1(-2\alpha-2j)=\lambda^{-2j-2\alpha+1}(\delta_{0j}^+)^{-1}A_j^-$$

which follows from (1.7), we find that for $0<r<b$

$$I_1(r)=\frac{\nu_0}{\mu\kappa_-}+\frac{\kappa_-}{\kappa_*\kappa_+}\sum_{j=0}^{\infty}\frac{j!A_j^-\sin\pi\alpha}{\Gamma(\alpha+j)}\left(\frac{r}{b}\right)^{2\alpha+2j-1}$$
$$I_2(r)=-\nu_0\kappa_*(\mu\kappa_-^2)^{-1}-2\kappa_+\nu_0(\pi\kappa_-)^{-1}\ln(r/a)-$$
$$-\frac{1}{\kappa_0}\sum_{j=0}^{\infty}\frac{j!A_j^-}{\Gamma(\alpha+j)}\left(\frac{r}{b}\right)^{2\alpha+2j-1}-\frac{1}{\kappa_0}\sum_{j=0}^{\infty}\frac{\Gamma(3/2-\alpha+j)}{\Gamma(1/2+j)}B_j\left(\frac{r}{b}\right)^{2j}$$

and finally obtain

$$\chi_1(r)=-\frac{2\kappa_+\nu_0}{\pi\kappa_*}\ln\frac{r}{a}-\frac{\kappa_-}{\kappa_*\kappa_0}\sum_{j=0}^{\infty}\frac{B_j\Gamma(3/2-\alpha+j)}{\Gamma(1/2+j)}\left(\frac{r}{b}\right)^{2j},\quad 0<r<b \tag{2.2}$$

Thus, the normal contact stresses at $r=0$ have a logarithmic singularity (the same kind of singularity is obtained if one considers friction-free contact between a wedge-shaped punch and a half-plane [4]).

Now let $b<r<a$. We have

$$I_1(r)=\frac{\nu_0}{\mu\kappa_-}-\frac{\nu_0\pi^{1/2}(r/a)^{2\alpha-1}F_*(\alpha,\alpha-1/2;\ r^2/a^2)}{\kappa_0\cos\pi\alpha\Gamma(1-\alpha)\Gamma(1/2+\alpha)}+$$
$$+\frac{2C}{\Gamma(1-\alpha)}\left(\frac{r}{a}\right)^{2\alpha-1}\left(1-\frac{r^2}{a^2}\right)^{-\alpha}-\frac{(r/a)^{2\alpha-1}}{\Gamma(1-\alpha)}\sum_{m=0}^{\infty}\frac{A_m^+}{m+1}\times$$
$$\times F_*\left(\alpha,m+1;\frac{r^2}{a^2}\right) \tag{2.3}$$
$$F_*(a,b;x)=b\sum_{j=0}^{\infty}\frac{(a)_j x^j}{(b+j)j!}$$

If $1/2<x<1$, the function $F_*(a,b;x)$ can be calculated using the following transformation formula for Gauss's function $F(a,b;b+1;x)$ [5]

$$F_*(a,b;x)=\frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(b+1-a)x^b}+\frac{b}{a-1}\sum_{j=0}^{\infty}\frac{(b+1-a)_j}{(2-a)_j}(1-x)^{j+1-a}$$

For $2^{1/2}b\leqslant r<a$, we can write $I_2(r)$ in the form

$$I_2(r)=-\frac{(b/r)^{3-2\alpha}}{\kappa_0\Gamma(\alpha-1)}\left[-\frac{a_0}{2}S\left(\frac{3}{2}-\alpha;\frac{b^2}{r^2}\right)+\right.$$
$$+\frac{a_0\ln\lambda-a_1}{3/2-\alpha}F_*\left(2-\alpha,\frac{3}{2}-\alpha;\frac{b^2}{r^2}\right)+\sum_{m=0}^{\infty}\frac{A_m^-}{m+1}F_*\left(2-\alpha,m+1;\frac{b^2}{r^2}\right)+$$
$$\left.+\sum_{m=0}^{\infty}\frac{B_m}{3/2-\alpha+m}F_*\left(2-\alpha,\frac{3}{2}-\alpha+m;\frac{b^2}{r^2}\right)\right] \tag{2.4}$$

$$S(\beta;\, x) = \sum_{j=0}^{\infty} \frac{(2-\alpha)_j x^j}{(\beta+j)^2 j!}$$

To calculate $I_2(r)$ for $b<r<\min\{2^{1/2}b, a\}$, we use (1.10) and the formula

$$S(\beta;\, x) = \Gamma(\beta)\Gamma(\alpha-1)[x^{\beta}\Gamma(\beta-1+\alpha)]^{-1}[\ln x - \psi(\beta) + \psi(\beta-1+\alpha)] -$$
$$- \frac{1}{1-\alpha} \sum_{j=1}^{\infty} \frac{(\beta-1+\alpha)_j}{(\alpha)_j} (1-x)^{j-1+\alpha}[\psi(\beta-1+\alpha+j) - \psi(\beta-1+\alpha)] \tag{2.5}$$
$$(\tfrac{1}{2} < x < 1)$$

which may be derived from [5, formula 7.4.1.5]. We have

$$I_2(r) = -\frac{(b/r)^{3-2\alpha}}{\kappa_0\Gamma(\alpha-1)} \left\{ -\frac{a_0}{2} S\left(\frac{3}{2}-\alpha;\, \frac{b^2}{r^2}\right) + (a_0\ln\lambda - a_1)H_0\left(\frac{3}{2}-\alpha;\, r\right) + \right.$$
$$\left. + \sum_{m=0}^{\infty} \left[A_m^- H_m(1;\, r) + B_m H_m\left(\frac{3}{2}-\alpha;\, r\right)\right]\right\} \tag{2.6}$$
$$H_m(\beta;\, r) = \frac{\Gamma(m+\beta)\Gamma(\alpha-1)}{\Gamma(m+\beta+\alpha-1)}\left(\frac{r}{b}\right)^{2m+2\beta} +$$
$$+ \frac{1}{1-\alpha}\sum_{j=1}^{\infty} \frac{(m+\beta+\alpha-1)_j}{(\alpha)_j}\left(1-\frac{b^2}{r^2}\right)^{j+\alpha-1}$$

The shear stresses are determined in similar fashion

$$\tau_{r\theta}(r,0) = -\nu_0\kappa_-\kappa_*^{-1} - r[b\kappa_*\Gamma(\alpha-1)]^{-1}\{\tfrac{1}{2}a_0 S(\tfrac{1}{2};\, r^2/b^2) +$$
$$+ (a_0\ln\lambda - a_1)G_0(\tfrac{1}{2};\, r) + \sum_{m=0}^{\infty}[A_m^- G_m(1-\alpha;\, r) + B_m G_m(\tfrac{1}{2};\, r)]\},\ 0<r<b \tag{2.7}$$
$$G_m(\beta;\, r) = \begin{cases} (\beta-m)^{-1}F_*(2-\alpha, \beta-m;\, r^2/b^2), & 0<r<2^{-1/2}b \\ \dfrac{\Gamma(-m+\beta)\Gamma(\alpha-1)}{\Gamma(-m+\alpha+\beta-1)}\left(\dfrac{r}{b}\right)^{2(m-\beta)} + \displaystyle\sum_{j=1}^{\infty}\dfrac{(-m+\alpha+\beta-1)_j}{(1-\alpha)(\alpha)_j} \times \end{cases}$$
$$\times \left(1-\frac{r^2}{b^2}\right)^{j+\alpha-1},\quad 2^{-1/2}b<r<b$$

It is obvious that the shear stresses vanish as zero: $\tau_{r\theta}(r,0)\to -\nu_0\kappa_-\kappa_*^{-1}$, $r\to 0$.

It can be shown that the contact stresses are continuous at b.

By (2.2)

$$\chi_1(b-0) = -\frac{2\kappa_+\nu_0}{\pi\kappa_*}\ln\lambda - \frac{\kappa_-}{\kappa_*\kappa_0}\sum_{j=0}^{\infty}\frac{B_j\Gamma(3/2-\alpha+j)}{\Gamma(1/2+j)}$$

On the other hand,

$$\chi_1(b+0) = I_1(b+0) + \kappa_*^{-1}\kappa_- I_2(b+0)$$

To compute $I_0(b+0)$ by formulae (2.6), we express A_m^- in terms of A_m^+ using the first equality of (1.7) and use the formula

$$S(\beta;\, 1) = \Gamma(\beta)\Gamma(\alpha-1)[\Gamma(\beta+\alpha-1)]^{-1}[\psi(\beta-1+\alpha) - \psi(\beta)]$$

which can be derived from (2.5). This yields the desired relationship

$$\lim_{r\to b-0}\sigma_\theta(r,0) = \lim_{r\to b+0}\sigma_\theta(r,0)$$

By (2.7) and the equality $\tau_{r\theta}(b+0,\, 0) = -\mu\chi_1(b+0)$, the shear stresses are also continuous across the friction/adhesion boundary.

In problem 1b, the conditions $L_a = 0$, or the relations (1.11) that follow from it, imply the

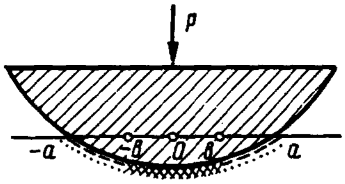

FIG. 2.

equalities $\sigma_\theta(a, 0) = \tau_{r\theta}(a, 0) = 0$. Indeed, letting $r \to a-0$ in (2.1), (2.3) and (2.4) and taking (1.7) into account, we obtain $\chi_1(a-0) = 0$.

3. THE DEPENDENCE OF λ ON THE PUNCH PROFILE

Let us consider problem Z_n, which is concerned with a punch whose profile is described by the function $f(x) = \Lambda|x|^n$ (where Λ is a given positive constant with the dimensions of x^{-n+1}), pressed down on the half-plane $|x| < \infty$, $y = 0$, by a symmetrically applied vertical force P (Fig. 2). The region of contact ($|x| < a$, $y = 0$) is divided into an adhesive region ($|x| < b$) and a region of Coulomb friction ($b < |x| < a$). The positions of the points a are determined *a posteriori*.

It was stated in [3] that the quantity $\lambda = b/a$ is independent of Λ, n and P, depending only on μ and ν; i.e. in a punch with profile $f(x) = \Lambda|x|^n (|\Lambda| < \infty, n \geq 1)$ or in the problem Z_0 of a flat-faced punch ($n = 0$) we will have the same λ. This conclusion is incorrect.

Let $p(x) = \sigma_y(x, 0)$, $q(x) = \tau_{xy}(x, 0)$. Then, using results obtained in [1], we obtain (for problem Z_n) a system of two singular integral equations

$$\kappa_- p(x) + \frac{2\kappa_+}{\pi} \int_0^a q(t) \frac{t\,dt}{t^2 - x^2} = 0, \quad 0 < x < b$$
$$-\kappa_- q(x) + \frac{2\kappa_+}{\pi} \int_0^a p(t) \frac{x\,dt}{t^2 - x^2} = \Lambda_n x^{n-1}, \quad 0 < x < a \tag{3.1}$$

where $\Lambda_n = n\nu_*^{-1}\Lambda$. In the frictional region $p(x)$ and $q(x)$ satisfy the condition

$$q(x) + \mu p(x) = 0 \quad (b \leqslant x < a)$$

but at $x = a$, as the punch profile is smooth in the neighbourhood of that point, we have

$$p(a) = q(a) = 0 \tag{3.2}$$

As the punch is in equilibrium

$$\int_0^a p(x)\,dx = -\frac{P}{2} \tag{3.3}$$

The differential operator

$$d_{nx} = \frac{n-1}{n} - \frac{x}{n}\frac{d}{dx}$$

(see [3]) makes the inhomogeneous system (3.1) homogeneous

$$\kappa_- p_0(x) + \frac{2\kappa_+}{\pi} \int_0^a q_0(t) \frac{t\,dt}{t^2 - x^2} = 0, \quad 0 < x < b$$
$$-\kappa_- q_0(x) + \frac{2\kappa_+}{\pi} \int_0^a p_0(t) \frac{x\,dt}{t^2 - x^2} = 0, \quad 0 < x < a \tag{3.4}$$

$$q_0(x) + \mu p_0(x) = 0, \quad (b < x < a) \tag{3.5}$$
$$p_0(x) = d_{nx} p(x), \quad q_0(x) = d_{nx} q(x)$$

where, by (3.2), $p(x)$ and $q(x)$ are expressed as follows in terms of $p_0(x)$ and $q_0(x)$

$$p(x) = nx^{n-1}\int_x^a \frac{p_0(t)}{t^n}\,dt, \quad q(x) = nx^{n-1}\int_x^a \frac{q_0(t)}{t^n}\,dt \tag{3.6}$$

In view of (3.3) and (3.6), we obtain

$$\int_0^a p_0(x) = -\frac{P}{2} \tag{3.7}$$

Thus, the system of integral equations (3.4), with conditions (3.5) and (3.7), is equivalent to the corresponding problem for a flat-faced punch, provided that, in addition

$$q(x) + \mu p(x) = nx^{n-1}\int_x^b \frac{q_0(t)+\mu p_0(t)}{t^n}\,dt \tag{3.8}$$

and to ensure the validity of the condition $q(b)+\mu p(b) = 0$, it is sufficient to require integrability of the function $q_0(x)+\mu p_0(x)$ in the vicinity of the point $x = b$.

The positions of b and a are defined by two equivalence conditions for systems (3.1) and (3.4). To determine these conditions, we consider the functions $\chi_1(x) = p_0(x) \in H^*[0, a)$ and $\chi_2(x) = q_0(x)+\mu p_0(x) \in H^*[0, b)$, where $H^*[0, c)$ is the space of functions that satisfy a Hölder condition in the interval $[0, c)$ and have an integrable singularity at $x = c$. Then $\chi_1(x)$, $\chi_2(x)$ is a solution of the system

$$\begin{aligned} &\kappa_-\chi_1(x) - \frac{2\mu\kappa_+}{\pi}\int_0^a \frac{\chi_1(t)t\,dt}{t^2-x^2} + \frac{2\kappa_+}{\pi}\int_0^b \frac{\chi_2(t)t\,dt}{t^2-x^2} = 0, \quad 0 < x < b \\ &\kappa_-\mu\chi_1(x) - \kappa_-\chi_2(x) + \frac{2\kappa_+}{\pi}\int_0^a \frac{\chi_1(t)x\,dt}{t^2-x^2} = 0, \quad 0 < x < a \end{aligned} \tag{3.9}$$

Extending system (3.9) to a semi-infinite interval by means of the functions $\psi_1(x)$, $\psi_2(x)$ ($\operatorname{supp}\psi_1(x) \subset (b, \infty)$, $\operatorname{supp}\psi_2(x) \subset (a, \infty)$), applying the Mellin transformation and using the notation (1.3), we obtain the homogeneous Riemann matrix problem (1.4) ($\nu_0 = 0$) whose solution is known [2]. Formulae for the solution are also obtained from (1.5) by setting $\nu_0 = 0$; they may be written as

$$\chi_1(a\tau) = C\chi_1^*(\tau), \quad \chi_2(b\tau) = C\chi_2^*(\tau), \quad C = Pa^{-1}C^*$$

(the asterisks mark quantities that are independent of a and P). Then the functions $p(x)$ and $q(x)$ are given by

$$p(a\xi) = Cp_n^*(\xi), \quad p_n^*(\xi) = n\xi^{n-1}\int_\xi^1 \chi_1^*(\tau)\frac{d\tau}{\tau^n}$$

$$q(a\xi) = Cq_n^*(\xi), \quad q_n^*(\xi) = n\xi^{n-1}\left[-\mu\int_\xi^1 \chi_1^*(\tau)\frac{d\tau}{\tau^n} + \lambda^{1-n}\int_{\xi/\lambda}^1 \chi_2^*(\tau)\frac{d\tau}{\tau^n}\right]$$

We require that the following two conditions hold

$$\begin{aligned} &\kappa_- p_n^*(\xi_0) + \frac{2\kappa_+}{\pi}\int_0^1 q_n^*(\tau)\frac{\tau\,d\tau}{\tau^2-\xi_0^2} = 0, \quad N_n^*(\xi_1)PC^* = -\Lambda_n a^n\xi_1^{n-1} \\ &N_n^*(\xi) = -\kappa_- q_n^*(\xi) + \frac{2\kappa_+\xi}{\pi}\int_0^1 p_n^*(\tau)\frac{d\tau}{\tau^2-\xi^2} \end{aligned} \tag{3.10}$$

(ξ_0, ξ_1 are arbitrary points in the intervals $(0, \lambda)$ and $(0, 1)$, respectively). Then systems (3.1) and (3.4) will be equivalent. The first condition of (3.10) is a transcendental equation for λ; λ is obviously a function of the parameters ν, μ and n, and, in problems with condition (3.2), is independent of P, E and Λ. To determine a, we use the second condition of (3.10)

$$a = [-PC^*\xi_1^{1-n}\Lambda_n^{-1}N_n^*(\xi_1)]^{1/n}$$

It was assumed in [3] that the shear displacements in the adhesive region were not zero but a polynomial

$$\left.\frac{\partial u}{\partial x}\right|_{y=0} = M_n|x|^{n-1}, \quad |x| < b, \quad M_n = n\nu_*^{-1}M \tag{3.11}$$

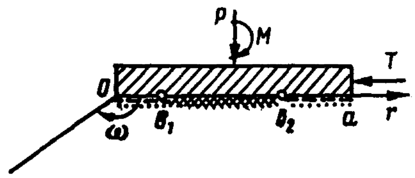

FIG. 3.

where the constant M is to be determined from condition (3.2). However, this condition is automatically satisfied thanks to (3.6). The condition for systems (3.1) and (3.4) to be equivalent, corresponding to the case (3.11), makes it impossible to choose M so that, for any n, λ is the same for problems Z_0 and Z_n. Thus, if condition (3.11) holds then, *a fortiori*, λ depends on the punch profile.

4. THE CONTACT PROBLEM FOR A WEDGE IN THE PRESENCE OF FRICTION AND COHESION

Let us consider a flat-faced punch ($0<r<a$, $\theta=0$) pressed into an elastic wedge ($0<r<\infty$, $-\omega<\theta<0$) by a vertical force P, a moment M and a horizontal force T (Fig. 3). The region of contact consists of an adhesive region ($b_1<r<b_2$) and frictional regions ($0<r<b_1$ and $b_2<r<a$); the boundary $\theta=-\omega$ is stress-free

$$\begin{aligned}
\theta=0:\quad & u_\theta=\delta_n+\gamma r,\quad 0<r<a;\quad u_r=\delta_t,\quad b_1<r<b_2,\\
& \tau_{r\theta}-\mu\sigma_\theta=0,\quad 0<r\leqslant b_1;\quad \tau_{r\theta}+\mu\sigma_\theta=0,\quad b_2\leqslant r<a\\
& \tau_{r\theta}=\sigma_\theta=0,\quad a<r<\infty\\
\theta=-\omega;\quad & \sigma_\theta=\tau_{r\theta}=0,\quad 0<r<\infty
\end{aligned}$$

(γ is the angle of rotation of the punch). If the conditions

$$\int_0^a \sigma_\theta(r,0)\,dr=-P,\quad \int_0^a \tau_{r\theta}(r,0)\,dr=-T,\quad \int_0^a \sigma_\theta(r,0)\,r\,dr=-M \tag{4.1}$$

are satisfied, the punch will be in equilibrium. Consider the following functions and their Mellin transforms

$$\begin{gathered}
\chi_1(r)=(\tau_{r\theta}-\mu\sigma_\theta)_{\theta=0},\quad \chi_2(r)=(\tau_{r\theta}+\mu\sigma_\theta)_{\theta=0}\\
\psi_1(r)=\frac{1}{\nu_*}\frac{\partial u_r}{\partial r}(r,0),\quad \psi_2(r)=\frac{1}{\nu_*}\frac{\partial u_\theta}{\partial r}(r,0)\\
\|\chi_{js},\psi_{js}\|=\int_0^\infty \|\chi_j(r),\psi_j(r)\|\,r^s dr
\end{gathered} \tag{4.2}$$

The functions ψ_{js} and χ_{js} satisfy the relationships

$$\begin{aligned}
& 2\mu\psi_{1s}=l_{21}(s)\,\chi_{1s}+l_{22}(s)\,\chi_{2s},\quad 2\mu\psi_{2s}=l_{11}(s)\,\chi_{1s}+l_{12}(s)\,\chi_{2s}\\
& l_{1j}(s)=-\mu\kappa_-+\kappa_+[2d(s)]^{-1}[(-1)^{j-1}(\sin 2\omega s+s\sin 2\omega)+2\mu s(s-1)\sin^2\omega]\\
& l_{2j}(s)=(-1)^j\kappa_-+\kappa_+[2d(s)]^{-1}[-\mu\sin 2\omega s+\mu s\sin 2\omega-2(-1)^j s(s+1)\sin^2\omega]
\end{aligned}$$

Putting $\lambda_1=b_1/a$, $\lambda_2=b_2/a$

$$\begin{aligned}
\Phi_1^-(s)&=\int_{\lambda_1}^1 \chi_1(ar)\,r^s dr,\quad \Phi_1^+(s)=\int_1^{1/\lambda_1}\chi_1(b_1 r)\,r^s dr\\
\Phi_2^-(s)&=\int_0^1 \chi_2(b_2 r)\,r^s dr,\quad \Phi_2^+(s)=2\mu\int_1^\infty \psi_2(ar)\,r^s dr
\end{aligned}$$

$$\Phi_3^-(s) = 2\mu \int_0^1 \psi_1(b_1 r) r^s dr, \quad \Phi_3^+(s) = 2\mu \int_1^\infty \psi_1(b_2 r) r^s dr \tag{4.3}$$
$$l(s) = l_{11}(s) l_{22}(s) - l_{12}(s) l_{21}(s) = 2\mu e(s)[d(s)]^{-1}$$
$$e(s) = \kappa_+^2 - s^2 \sin^2\omega - \kappa \sin^2 \omega s$$

we obtain a Riemann matrix problem [2] which, after factorizing the functions

$$k_j(s) = L_j^+(s) L_j^-(s) X_j^+(s)[X_j^-(s)]^{-1}, \quad s \in \Gamma \quad (j = 0, 1, 2)$$
$$k_0(s) = l_{11}(s), \quad k_n(s) = l(s)[l_{1n}(s)]^{-1} \quad (n = 1, 2)$$
$$L_0^+(s) = \frac{\kappa_0 \Gamma(-\delta s)}{\Gamma(1-\alpha-\delta s)}, \quad L_0^-(s) = \frac{\Gamma(1+\delta s)}{\Gamma(\alpha+\delta s)}, \quad \delta = \frac{\omega}{\pi}, \quad \kappa_0 = \frac{\kappa_+}{\sin \pi\alpha}$$
$$L_1^+(s) = -\frac{\kappa_1 \Gamma(-\delta s)\Gamma(1-\alpha-\delta s)}{\Gamma^2(\frac{1}{2}-\delta s)}, \quad L_1^-(s) = \frac{\Gamma(1+\delta s)\Gamma(\alpha+\delta s)}{\Gamma^2(\frac{1}{2}+\delta s)}, \quad \kappa_1 = \frac{2\mu\kappa}{\kappa_0}$$
$$L_2^+(s) = \frac{\kappa_1 \Gamma(-\delta s)\Gamma(\alpha-\delta s)}{\Gamma^2(\frac{1}{2}-\delta s)}, \quad L_2^-(s) = \frac{\Gamma(1+\delta s)\Gamma(1-\alpha+\delta s)}{\Gamma^2(\frac{1}{2}+\delta s)}$$
$$X_j(s) = \exp\left(\frac{1}{2\pi i} \int_\Gamma \frac{\ln k_j^0(s)}{s-z} ds\right), \quad \operatorname{ind} k_j^0(s) = 0$$

$$k_0^0(s) = l_{11}^0(s), \quad k_n^0(s) = l^0(s)[l_{1n}^0(s)]^{-1} \quad (n = 1, 2)$$
$$l_{1n}^0(s) = [-\mu\kappa_- + (-1)^n \kappa_+ \operatorname{ctg} \omega s]^{-1} l_{1n}(s), \quad l^0(s) = -(2\mu\kappa)^{-1} \operatorname{tg}^2 \omega s\, l(s)$$

may be rewritten in the form

$$\frac{\Phi_2^+(s) + (s+1)^{-1} C_0}{L_0^+(s) X_0^+(s)} = \frac{L_0^-(s)}{X_0^-(s)} \Phi_1^-(s) + \lambda_2^{s+1} \frac{l_{12}(s)}{l_{11}(s)} \frac{L_0^-(s)\Phi_2^-(s)}{X_0^-(s)}, \quad C_0 = \frac{2\mu\gamma}{\nu_*}$$
$$\frac{\Phi_3^+(s)}{L_1^+(s) X_1^+(s)} - \lambda_2^{-s-1} \frac{l_{21}(s)}{l_{11}(s)} \frac{\Phi_2^+(s) + (s+1)^{-1} C_0}{L_1^+(s) X_1^+(s)} = \frac{L_1^-(s)}{X_1^-(s)} \Phi_2^-(s) -$$
$$- (\lambda_1/\lambda_2)^{s+1} l_{11}(s)[l(s) X_1^-(s)]^{-1} L_1^-(s) \Phi_3^-(s)$$

$$-[L_2^-(s)]^{-1} X_2^-(s) \Phi_3^-(s) = L_2^+(s) X_2^+(s) \Phi_1^+(s) + (\lambda_2/\lambda_1)^{s+1} l_{12}(s)[l(s)]^{-1} \times$$
$$\times L_2^+(s) X_2^+(s) \Phi_3^+(s) - \lambda_1^{-s-1} l_{22}(s)[l(s)]^{-1} L_2^+(s) X_2^+(s) [\Phi_2^+(s) + (s+1)^{-1} C_0] \tag{4.4}$$

Let σ_j^- ($j = 0, 1, \ldots$) denote the poles of $l_{12}(s)[l_{11}(s)]^{-1}$ in D^-, and σ_j^+ ($j = 0, 1, \ldots$) those of $l_{21}(s)[l_{11}(s)]^{-1}$ ($\sigma_j^+ \in D^+$). All the numbers σ_j^+ are real. Let s_j ($j = 0, 1, \ldots$) be the (complex-valued) roots of the function $e(s)$ defined in (4.3). Then $l_{11}(s)[l(s)]^{-1}$ has poles in D^- at the points $s = s_j$, and the functions $l_{12}(s)[l(s)]^{-1}$, $l_{22}(s)[l(s)]^{-1}$ have poles in D^+ at the points $s = s_j$. Following the scheme of [2], we define

$$\Psi_1^\pm(s) = \sum_{j=0}^\infty \frac{A_j^\pm}{s - \sigma_j^\mp}, \quad \Psi_2^\pm(s) = \sum_{j=0}^\infty \frac{B_j^\pm}{s \mp s_j}$$

($A_j^\pm$, $B_j^\pm$ are coefficients, as yet unknown), and we obtain a solution of problem (4.4)

$$\Phi_1^-(s) = \frac{X_0^-(s)}{L_0^-(s)} \Omega_1(s) - \lambda_2^{s+1} \frac{l_{12}(s) X_1^-(s)}{l_{11}(s) L_1^-(s)} \Omega_2(s) + \lambda_1^{s+1} \frac{l_{12}(s) L_2^-(s)}{l(s) X_2^-(s)} \Omega_3(s)$$
$$\Phi_1^+(s) = \lambda_1^{-s-1} \Phi_1^-(s), \quad \Phi_2^-(s) = \frac{X_1^-(s)}{L_1^-(s)} \Omega_2(s) - \left(\frac{\lambda_1}{\lambda_2}\right)^{s+1} \frac{l_{11}(s) L_2^-(s)}{l(s) X_2^-(s)} \Omega_3(s)$$
$$\Phi_2^+(s) = -(s+1)^{-1} C_0 + L_0^+(s) X_0^+(s) \Omega_1(s), \quad \Phi_3^-(s) = -L_2^-(s)[X_2^-(s)]^{-1} \Omega_3(s) \tag{4.5}$$
$$\Phi_3^+(s) = L_1^+(s) X_1^+(s) \Omega_2(s) + \lambda_2^{-s-1} l_{21}(s)[l_{11}(s)]^{-1} L_0^+(s) X_0^+(s) \Omega_1(s)$$

$$\Omega_1(s) = (s+1)^{-1}\nu_0 C_0 + C_1 + \Psi_1^+(s)$$
$$\Omega_2(s) = -(s+1)^{-1}\nu_1 C_0 + C_2 + \Psi_1^-(s) + \Psi_2^+(s), \quad \Omega_3(s) = -(s+1)^{-1}\nu_2 C_0 + \Psi_2^-(s)$$
$$\nu_0 = [L_0^+(-1)X_0^+(-1)]^{-1}$$
$$\nu_1 = \frac{1}{X_1^+(-1)L_1^+(-1)} \lim_{s\to -1} \frac{l_{21}(s)}{l_{11}(s)}, \quad \nu_2 = L_2^+(-1)X_2^+(-1) \lim_{s\to -1} \frac{l_{22}(s)}{l(s)}$$

(if $\omega \neq \pi$ or $\omega \neq \operatorname{arcctg}\mu$, then $\nu_1 = 0$; if $\sin^2\omega \neq 1-\nu$, then $\nu_2 = 0$); C_1, C_2 are arbitrary constants.

To determine the boundaries of the adhesive and frictional regions, we determine the stress intensity factors

$$K_1 = \lim_{r\to b_1+0} (r-b_1)^{1-\alpha}\chi_1(r) = \frac{b_1^{1-\alpha}(\nu_2 C_0 - B_*)}{\delta^{\alpha-1}\kappa_1\Gamma(\alpha)}, \quad B_* = \sum_{j=0}^{\infty} B_j^-$$
$$K_2 = \lim_{r\to b_2-0} (b_2-r)^{1-\alpha}\chi_2(r) = C_2 b_2^{1-\alpha}[\Gamma(\alpha)\delta^\alpha]^{-1}$$

and stipulate that $K_1 = 0$, $K_2 = 0$. Then

$$C_2 = 0, \quad \nu_2 C_0 - B_* = 0$$

We express the coefficients $A_n^\pm$, $B_n^\pm$ in the form

$$A_n^\pm = \sum_{k=0}^{1} C_k A_{nk}^\pm, \quad B_n^\pm = \sum_{k=0}^{1} C_k B_{nk}^\pm$$

Then a necessary and sufficient condition for the functions $\Phi_j^\pm(s)$ ($j = 1, 2, 3$) defined in (4.5) to be analytic in $D^\pm$ is that

$$A_{nk}^- = \lambda_2^{-\sigma_n^+-1} r_n^{(1)} \left(-\frac{\nu_0\delta_{k0}}{\sigma_n^+ + 1} - \delta_{k1} + \sum_{j=0}^{\infty} \frac{A_{jk}^+}{\sigma_j^- - \sigma_n^+}\right)$$
$$B_{nk}^+ = \left(\frac{\lambda_1}{\lambda_2}\right)^{s_n+1} r_n^{(3)} \left(-\frac{\nu_2\delta_{k0}}{s_n+1} + \sum_{j=0}^{\infty} \frac{B_{jk}^-}{s_n+s_j}\right)$$
$$A_{nk}^+ = \lambda_2^{\sigma_n^-+1} r_n^{(2)} \left[-\frac{\nu_1\delta_{k0}}{\sigma_n^- + 1} + \delta_{k2} + \sum_{j=0}^{\infty}\left(\frac{A_{jk}^-}{\sigma_n^- - \sigma_j^+} + \frac{B_{jk}^+}{\sigma_n^- - s_j}\right)\right]$$
$$B_{nk}^- = \left(\frac{\lambda_1}{\lambda_2}\right)^{s_n-1} r_n^{(4)} \left[\frac{\nu_1\delta_{k0}}{s_n-1} + \delta_{k2} - \sum_{j=0}^{\infty}\left(\frac{A_{jk}^-}{s_n+\sigma_j^+} + \frac{B_{jk}^+}{s_n+s_j}\right)\right]$$
$$(n = 0, 1, \ldots;\ k = 0, 1) \tag{4.6}$$
$$r_n^{(1)} = \frac{l_{21}(s)}{l_{11}'(s)} \frac{L_0^+(s)X_0^+(s)}{L_1^+(s)X_1^+(s)}\bigg|_{s=\sigma_n^+}, \quad r_n^{(2)} = \frac{l_{12}(s)}{l_{11}'(s)} \frac{L_0^-(s)X_1^-(s)}{X_0^-(s)L_1^-(s)}\bigg|_{s=\sigma_n^-}$$
$$r_n^{(3)} = \frac{l_{11}(s)}{l'(s)} \frac{L_1^-(s)L_2^-(s)}{X_1^-(s)X_2^-(s)}\bigg|_{s=s_n}, \quad r_n^{(4)} = \frac{l_{12}(s)}{l'(s)} L_1^+(s)L_2^+(s)X_1^+(s)X_2^+(s)\big|_{s=-s_n}$$

(conditions (4.6) correspond to an infinite algebraic system of normal type).

Assuming that the three equilibrium conditions (4.1) are satisfied, we obtain a formula for the angle of rotation of the punch

$$\gamma = -\nu_* a^{-1} P F_1 [a_{00}F_1 + a_{01}(\nu_2 - F_0)]^{-1} \tag{4.7}$$

and a system of two transcendental equations for λ_1 and λ_2

$$f_0[a_{j0}F_1 + a_{j1}(\nu_2 - F_0)] = [a_{00}F_1 + a_{01}(\nu_2 - F_0)]f_j \quad (j = 1, 2)$$

We have used the following notation here and in (4.7)

$$F_k = \sum_{j=0}^{\infty} B_{jk}^- \ (k=0,1); \quad f_0 = \frac{2\mu P}{a}, \quad f_1 = \frac{2T}{a}, \quad f_2 = \frac{2\mu M}{a^2}$$

$$a_{jk} = \lambda_2 e_1 [1 + (-1)^j d_0](-\nu_1 \delta_{k0} + \omega_{k0}^- + \chi_{k0}^+) - (-1)^j e_0(\nu_0 \delta_{k0} + \delta_{k1} + \omega_{k0}^+)$$

$$a_{2k} = -q_0 (\tfrac{1}{2}\nu_0 \delta_{k0} + \delta_{k1} + \omega_{k1}^+ \ (j,k=0,1),$$

$$d_0 = \frac{2\omega + \sin 2\omega + 2\mu \sin^2 \omega}{-2\omega - \sin 2\omega + 2\mu \sin^2 \omega}, \quad g_0 = \frac{X_0^-(1)}{L_0^-(1)}, \quad e_j = \frac{X_j^-(0)}{L_j^-(0)} \quad (j=0,1)$$

$$\omega_{km}^{\pm} = \sum_{j=0}^{\infty} \frac{A_{jk}^{\pm}}{m - \sigma_j^{\pm}}, \quad \chi_{k0}^+ = -\sum_{j=0}^{\infty} \frac{B_{jk}^+}{s_j}$$

We will now determine the singularities of the functions $\tau_{r\theta}$, σ_θ, $\partial u_r/\partial r$, $\partial u_\theta/\partial r$ as $r \to 0$ $(\theta = 0)$. Using (4.2), (4.3) and (4.5), we obtain

$$\chi_2(b_2 r) = \frac{1}{2\pi i} \int_\Gamma \left\{ \frac{l_{11}(s)}{l(s)} L_1^+(s) X_1^+(s) \Omega_2(s) - \left(\frac{\lambda_1}{\lambda_2}\right)^{s+1} \frac{l_{11}(s)}{l_{12}(s)} \frac{\Omega_3(s)}{L_2^+(s) X_2^+(s)} \right\} \frac{ds}{r^{s+1}}$$

By Cauchy's Theorem, in view of (4.6), we obtain

$$\chi_2(r) = -\sum_{n=0}^{\infty} \left\{ \frac{l_{11}(s)\Omega_3(s)}{l'_{12}(s) X_2^+(s) L_2^+(s)} \right\}_{s=\beta_n^+} \left(\frac{r}{b_1}\right)^{-\beta_n^+ - 1} \tag{4.8}$$

where β_n^+ $(n = 0, 1, \ldots)$ are the roots of $l_{12}(s)$ in D^+. Similarly, we have

$$2\mu \eta_1(r) = \sum_{n=0}^{\infty} \left\{ \frac{l(s)\Omega_3(s)}{l'_{12}(s) L_2^+(s) X_2^+(s)} \right\}_{s=\beta_n^+} \left(\frac{r}{b_1}\right)^{-\beta_n^+ - 1}$$

Thus, the contact stresses and radial derivatives of the displacements behave like r^σ as $r \to 0$ $(\sigma = -\beta_0^+ - 1$, β_0^+ has the largest real part of all roots $\beta_n^+)$.

5. NUMERICAL IMPLEMENTATION

Problems 1a and 1b have been worked out numerically for $\nu = 0.3$ and $E/P = 1$. Below we present the computed values of $\lambda \times 10^3$ for a flat-faced punch, for different values of μ (the figures in the second row are taken from [1]; those in the third were computed specially for this paper, for problem 1a with $2\gamma = \pi$)

μ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$\lambda \times 10^3$	36.9	366	695	868	942	974	989	993
$\lambda \times 10^3$	36.5	360	689	865	941	973	988	992

(these figures are independent of E/P). Table 1 lists values of $\lambda \times 10^3$ for some values of μ and $\gamma_0 = \pi - \gamma$ with $E/P = 1$, in problem 1a. For problem 1b, as remarked in Sec. 3, λ is independent of the angle γ and quotient E/P, depending only on μ and ν. Here are the values of λ and a for a few values of μ

μ	0.1	0.3	0.5	0.7	0.9	1.1	1.3
λ	3.18×10^{-4}	0.109	0.289	0.413	0.497	0.556	0.601
λ	10.27	10.03	9.84	9.70	9.59	9.50	9.43

(the lower row corresponds to the case $\gamma_0 = 3°$), as well as a for fixed $\mu = 0.3$ and a few values of γ_0

γ_0	1°	5°	10°	15°
a	50.28	10.03	4.98	3.28

Figure 4 shows plots of the contact stresses for $\mu = 0.3$. Curves 1 and 1° correspond to normal stresses $-P^{-1}\sigma_\theta(ar, 0)$ in problem 1a with $\gamma_0 = 5°$ and $\gamma_0 = 0$ (a flat-faced punch), and curves 2 and 2° correspond to stresses $P^{-1}\tau_{r\theta}(ar, 0)$ in the same cases. In problem 1b, for the case $\gamma_0 = 5°$, plots of the stresses $-P^{-1}\sigma_\theta(r, 0)$ and $P^{-1}\tau_{r\theta}(r, 0)$ are shown in Fig. 5, where curves 1 and 2 correspond to normal and shear stresses with $\mu = 0.3$, curves 1° and 2° correspond to the same stresses with $\mu = 0.7$. Plots of the functions σ_y and $\tau_{xy}(\mu\sigma_y)^{-1}$

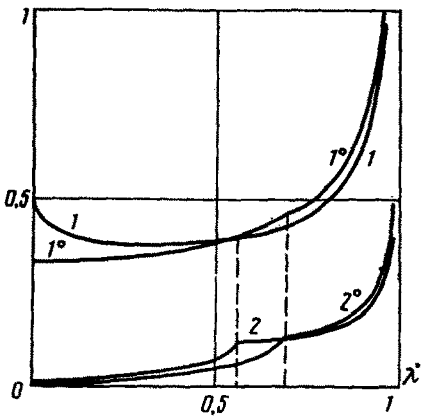

FIG. 4.

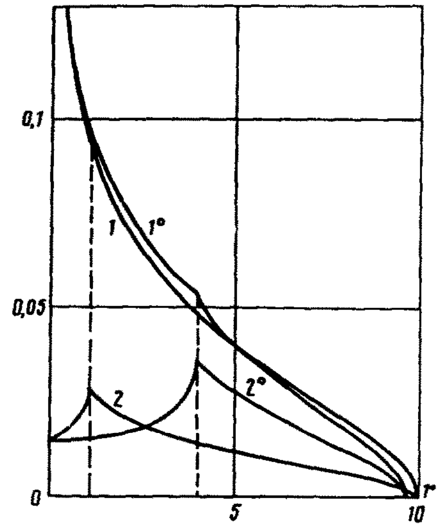

FIG. 5.

TABLE 1

γ_0	$\mu = 0.1$	0.3	0.5	0.7	0.9
1°	29.0	662	930	986	–
5°	11.4	558	874	966	992
10°	4.09	442	779	917	970
15°	1.91	345	676	840	922

for $y = 0$ in problem 1a with $\gamma_0 = 0$ (flat punch), $\nu = 0$, $\mu = 0.3683$ (in which case $\lambda = 0.3$) are in good agreement with the corresponding curves in [3], which were based on numerical computations.

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